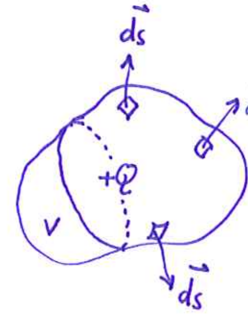
Gauss's Law

Maxwell's first equation was: $\boxed{\vec{\nabla}\cdot\vec{D}=\rho_v \quad \text{(Gauss's law)}}$ (Differential form)

If we integrate it: $\int_V \vec{\nabla}\cdot\vec{D}\,dv = \int_V \rho_v\,dv = Q$



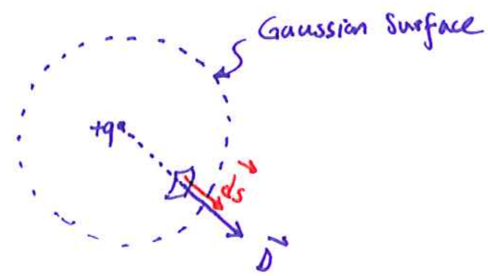
we also know from divergence theorem that: $\int_V \vec{\nabla}\cdot\vec{D}\,dv = \oint_S \vec{D}\cdot\vec{ds}$

we can therefore write the integral form of Gauss's law as:

$$\boxed{\oint_S \vec{D}\cdot\vec{ds} = Q}$$ (Gauss's law - integral form)

The surface S that encloses charge Q, is called Gaussian surface.

Example Derive Coulomb's law for a charge q using Gauss's law:



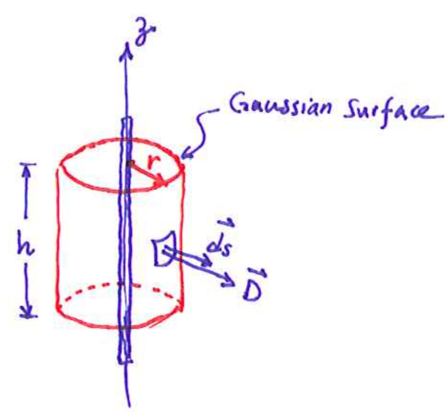
$\oint_S \vec{D}\cdot\vec{ds} = q$

$\left.\begin{array}{l}\vec{D} = D\,\hat{R} \\ \vec{ds} = ds\,\hat{R}\end{array}\right\} \vec{D}\cdot\vec{ds} = D\,ds\,\overbrace{\hat{R}\cdot\hat{R}}^{1} = D\,ds$

$\int_S \vec{D}\cdot\vec{ds} = D\int_S ds = D(4\pi R^2) = q \rightarrow \vec{D} = \frac{q}{4\pi R^2}\hat{R}$

$\vec{E} = \frac{\vec{D}}{\varepsilon} \rightarrow \vec{E} = \frac{q}{4\pi\varepsilon R^2}\hat{R}$

Example Find the electric of an infinite line of charge.



$\int_S \vec{D}\cdot\vec{ds} = Q$

$\vec{D} = D\hat{r}$

$\left\{\begin{array}{l}\vec{ds} = ds\,\hat{r} \text{ on the sides} \\ \vec{ds} = ds\,\hat{z} \text{ or } -ds\,\hat{z} \text{ at top and bottom}\end{array}\right.$

$Q = \rho_\ell h$

$$\int_S \vec{D}\cdot\vec{ds} = \int_{\substack{\text{side}\\\text{surface}}} \vec{D}\cdot\vec{ds} + \int_{\substack{\text{top}\\\text{surface}}} \vec{D}\cdot\vec{ds} + \int_{\text{Bottom Surface}} \vec{D}\cdot\vec{ds} = \int D\, ds\, \overbrace{\hat{r}\cdot\hat{r}}^{=1} + \int D\, ds\, \overbrace{\hat{r}\cdot\hat{z}}^{=0} + \int D\, ds\, \overbrace{(-\hat{r}\cdot\hat{z})}^{=0}$$

$$\rightarrow \int_S \vec{D}\cdot\vec{ds} = \int_{\substack{\text{side}\\\text{surface}}} D\, ds = D \int_{\substack{\text{side}\\\text{surface}}} ds = D\,(2\pi r h) \rightarrow D(2\pi r h) = \rho_l h$$

$$\rightarrow \vec{D} = \frac{\rho_l}{2\pi r}\hat{r} \rightarrow \boxed{\vec{E} = \frac{\rho_l}{2\pi\varepsilon r}\hat{r}}$$ Infinite line of charge

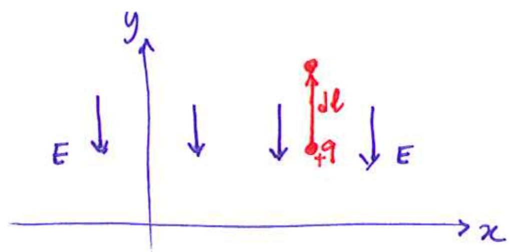
Electric Scalar Potential

In electric circuit we work with voltage and current. The voltage V is the amount of work (or potential) required to move a unit charge from one point to the other.

voltage = voltage potential = electric potential

The voltage is a result of the existance of the electric field (E). We want to find the relation between voltage V and electric field E. Consider a unifor field E. The force on a charge q in this field is $\vec{F}_e = q\vec{E}$. If we want to move the charge against this force we have to apply a force of at least $\vec{F}_{ext} = -q\vec{E}$. The work done to move the charge by l distance is therefore: $dw = \vec{F}_{ext}\cdot\vec{dl} = -q\vec{E}\cdot\vec{dl}$ (J)

$$\left.\begin{array}{l}\vec{E} = E(-\hat{y}) \rightarrow dw = qE\,\hat{y}\cdot\vec{dl} \\ \vec{dl} = dl\,\hat{y}\end{array}\right\} dw = qE\,dy$$



The differential electric potential (or differential voltage) dv is:

$$dv = \frac{dw}{q} = \frac{1}{q}(-q\vec{E}\cdot\vec{dl}) = -\vec{E}\cdot\vec{dl} \quad \left(\frac{J}{C} \text{ or } V\right)$$

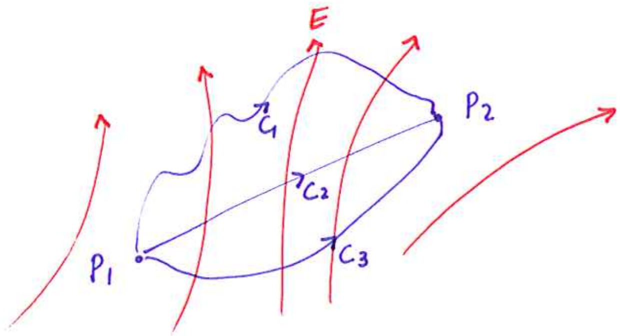
The potential difference between two points P_1 and P_2 is then:

$$V_{21} = \int_{P_1}^{P_2} dV = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E}\cdot\vec{dl} \rightarrow \boxed{V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} E\cdot dl}$$

Kirchhoff's voltage law

The net voltage drop around a closed loop is zero.

Moreover, the potential difference doesn't depend on the path from P_1 to P_2.



This means that the line integral of $\vec{E}$ around any closed loop is zero:

$$\boxed{\oint_C \vec{E}\cdot d\vec{l} = 0}$$

Electric Potential due to point charges

From Coulomb's law we have: $\vec{E} = \hat{R}\,\dfrac{q}{4\pi\varepsilon R^2}$

$$V = -\int \vec{E}\cdot d\vec{l}$$

$$d\vec{l} = \hat{R}\,dR \rightarrow V = -\int_{-\infty}^{R} \frac{q}{4\pi\varepsilon R^2}\,dR\,(\overbrace{\hat{R}\cdot\hat{R}}^{1}) = \frac{q}{4\pi\varepsilon R}$$

If q is not at origin and is at location $\vec{R}_1$:

$$\boxed{V = \frac{q}{4\pi\varepsilon\,|R-R_1|}} \quad (V)$$

If we have N charges of $q_1, q_2, \cdots q_N$ at locations $R_1, R_2, \cdots R_N$:

$$\boxed{V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|R-R_i|} \quad (V)}$$

Electric Potential due to Continous Distribution

Similar to our discussion for electric fied $\vec{E}$ for continous charge distribution we have:

$$\boxed{\begin{aligned} V &= \frac{1}{4\pi\varepsilon}\int_v \frac{\rho_v dv}{R} \quad \text{(volume distribution)} \\ V &= \frac{1}{4\pi\varepsilon}\int \frac{\rho_s ds}{R} \quad \text{(surface distribution)} \\ V &= \frac{1}{4\pi\varepsilon}\int \frac{\rho_l dl}{R} \quad \text{(line distribution)} \end{aligned}}$$

Electric field as a function of Electric Potential

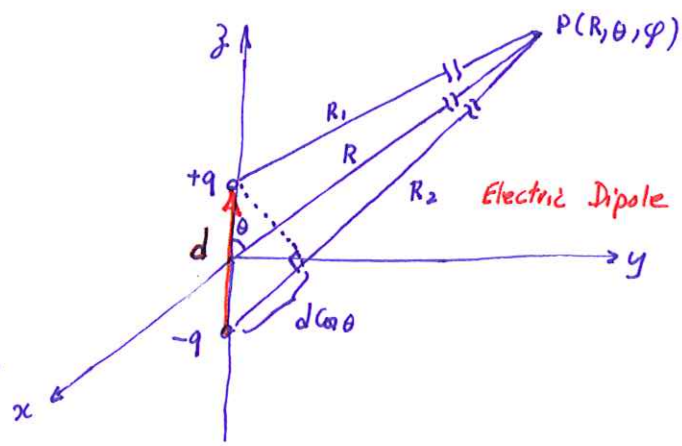
we derived: $dv = -\vec{E}\cdot\vec{dl}$

for a scalar function we have: $dv = \vec{\nabla}V\cdot\vec{dl}$

$$\boxed{\vec{E} = -\vec{\nabla}V}$$

Example

Electric field of an electric dipole.

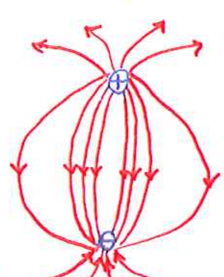


V at point $P(R,\theta,\phi)$ is:

$$V = \frac{q}{4\pi\varepsilon_0 R_1} + \frac{-q}{4\pi\varepsilon_0 R_2} = \frac{q}{4\pi\varepsilon_0}\,\frac{R_2 - R_1}{R_1 R_2}$$

if $d \ll R$, R_1 and R_2 lines are appoximately parallel and we have:

$$\left.\begin{aligned} R_2 - R_1 &\simeq d\cos\theta \\ R_1 R_2 &\simeq R^2 \end{aligned}\right\} \quad V = \frac{q d \cos\theta}{4\pi\varepsilon_0 R^2}$$



Electric field pattern

$$q d\cos\theta = q\,\vec{d}\cdot\hat{R} = \vec{P}\cdot\hat{R}$$

where we defined $\boxed{\vec{P} = q\vec{d} \quad \text{dipole moment}}$

$$\rightarrow \quad \boxed{V = \frac{\vec{P}\cdot\hat{R}}{4\pi\varepsilon_0 R^2}} \quad \text{Electric Dipole when } R \gg d$$

In spherical coordinate we have:

$$\vec{E} = -\vec{\nabla} V$$

$$= -\left(\hat{R} \frac{\partial V}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{R \sin\theta} \frac{\partial V}{\partial \varphi} \right)$$

if we substitude $V = \dfrac{qd\cos\theta}{4\pi\varepsilon_0 R^2}$ and make the derivatives we get:

$$\boxed{\vec{E} = \frac{qd}{4\pi\varepsilon_0 R^3} \left(\hat{R}\, 2\cos\theta + \hat{\theta} \sin\theta \right) \quad \left(\frac{V}{m}\right)}$$ when $R \gg d$

Poisson's Equation

we had $\vec{\nabla} \cdot \vec{D} = \rho_v \implies \vec{\nabla} \cdot (\varepsilon E) = \rho_v \rightarrow \vec{\nabla} \cdot \vec{E} = \dfrac{\rho_v}{\varepsilon}$

we also had $\vec{E} = -\vec{\nabla} V \implies \vec{\nabla} \cdot (-\vec{\nabla} V) = \dfrac{\rho_v}{\varepsilon} \rightarrow \vec{\nabla} \cdot (\vec{\nabla} V) = -\dfrac{\rho_v}{\varepsilon}$

This is Laplacian: $\nabla^2 V = \vec{\nabla} \cdot (\vec{\nabla} V) = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}$

So we can write: $\boxed{\nabla^2 V = -\dfrac{\rho_v}{\varepsilon} \quad \text{(poisson's equation)}}$

If there is no free charge in the medium $\rho_v = 0 \implies$

$$\boxed{\nabla^2 V = 0 \quad \text{(laplace's equation)}}$$

Poisson's and Laplace's equations are useful to find the potential in a region where the value of potential at the boundaries is known.

Electrical Properties of Materials

Electromagnetic constitune parameters are:

ε : electrical permitivity

μ : magnetic permeability

σ : electrical conductivity

Homogeneous material: their parameters don't change from point to point.

Isotropic material: these parameters are independent of direction.

In this course we assume all materials are homogeneous and isotropic.

Materials are classified as Conductors (metals) or dielectrics (insulators) based on the magnitudes of their conductivity σ.

In zero external electric field, electrons have random movement in material and the current is zero. upon applyin an external field, their movement takes a direction characterized by an average velocity called drift velocity (u_e), that gives rise to a conduction current.

Perfect conductor: $\sigma = 0$ typical metals $\sigma = 10^{6} - 10^{7}$ S/m

Perfect dielectric: $\sigma = \infty$ typical dielectrics $\sigma = 10^{-10} - 10^{-17}$ S/m

Semiconductors: σ is in between metals and dielectrics.

For example Ge has $\sigma = 2.2$ S/m.

Conductors

Drift velocity of electrons in a conducting material is related to the externally applied electric field:

$$\underbrace{\vec{u}_e}_{\text{Drift velocity}} = -\underbrace{\mu_e}_{\text{electron mobility}} \underbrace{\vec{E}}_{\text{Electric field}} \quad (m/s) \rightarrow \text{For electrons}$$

$$\vec{u}_h = \underbrace{\mu_h}_{\text{hole mobility}} \vec{E} \quad (m/s) \rightarrow \text{For holes}$$

Current density in a volume density of charge ρ_v moving with a velocity $\vec{u}$ is:

$$\vec{J} = \rho_v \vec{u}$$

The current density may consist of components from both electrons J_e and holed J_h:

$$\vec{J} = \vec{J}_e + \vec{J}_h = \rho_{ve}\vec{u}_e + \rho_{vh}\vec{u}_h \quad (A/m^2)$$

using $\vec{U} = \mu\vec{E} \Rightarrow \quad \vec{J} = (-\rho_{ve}\mu_e + \rho_{vh}\mu_h)\vec{E}$

where $\underbrace{\rho_{ve} = -N_e e}_{\text{\# electrons per unit volume}}$ and $\underbrace{\rho_{vh} = N_h e}_{\text{\# of holes per unit volume}}$ where $e = 1.6 \times 10^{-19}$ C

Conductivity is defined as:

$$\boxed{\sigma = eN_e\mu_e + eN_h\mu_h} \quad (S/m) \quad (\text{Semiconductor})$$

$$= -\rho_{ve}\mu_e + \rho_{vh}\mu_h$$

For a good conductor usually $N_h\mu_h \ll N_e\mu_e$ (or $N_e\mu_e \ll N_h\mu_h$) $\Rightarrow$

$$\boxed{\sigma = eN_e\mu_e} \quad (S/m) \quad (\text{conductor})$$

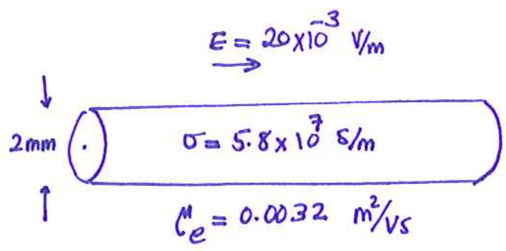
In either case : $\boxed{\vec{J} = \sigma\vec{E}}$ (A/m^2) (Ohm's law — point form)

Perfect dielectric $\sigma = 0 \rightarrow \vec{J} = 0$

Perfect conductor $\sigma = \infty \rightarrow \vec{E} = 0$

The electric potential in a conductor is constant everywhere, which we call it equipotential medium: $V_{21} = -\int_1^2 \vec{E}\cdot d\vec{l}$ since $\vec{E} = 0$ inside the conductor $\Rightarrow V_{21} = 0 \rightarrow V_2 = V_1$

Example



The conductor shown in the picture is applied to an electric field of $20 \frac{mV}{m}$.

Find (a) volume charge density ρ_v of free electrons?
(b) current density J?
(c) the current flowing in the wire?
(d) the electron drift velocity
(e) volume density of free electrons? N_e

Answer:

(a) $$\rho_{ve} = -\frac{\sigma}{\mu_e} = -\frac{5.8\times10^{7}}{0.0032} = -1.81\times10^{10}\ (C/m^3)$$

(b) $$\vec{J} = \sigma\vec{E} = 5.8\times10^{7}\times 20\times10^{-3} = 1.16\times10^{6}\ (A/m^2)$$

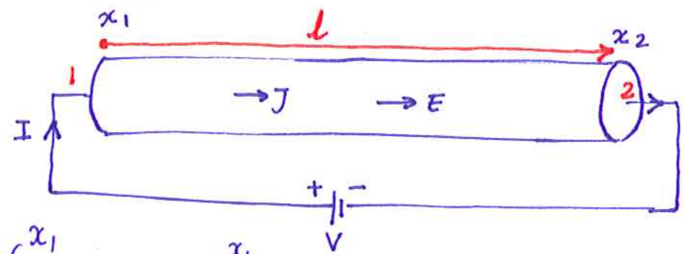
(c) $$\vec{I} = \vec{J}A = J\left(\frac{\pi d^2}{4}\right) = 1.16\times10^{6}\left(\frac{\pi\times4\times10^{-6}}{4}\right) = 3.64A$$

(d) $$\vec{u}_e = -\mu_e\vec{E} = -0.0032\times20\times10^{-3} = -6.4\times10^{-5}\ m/s$$

(e) $$N_e = -\frac{\rho_{ve}}{e} = \frac{1.81\times10^{10}}{1.6\times10^{-19}} = 1.13\times10^{29}\ \text{electron}/m^3$$

Resistance

Using the point form of ohm's law, we can derive an expression for the resistance R of a conductor of length l and uniform section A.



$$V = V_1 - V_2 = -\int_{x_2}^{x_1} E\cdot dl = -\int_{x_2}^{x_1} \hat{x}E_x\cdot\hat{x}\,dl = E_x l$$

$$I = \int_A \vec{J}\cdot\vec{ds} = \int_A \sigma\vec{E}\cdot\vec{ds} = \sigma E_x A$$

$$R = \frac{V}{I} = \frac{E_x l}{\sigma E_x A} = \frac{1}{\sigma}\frac{l}{A}\quad(\Omega)$$

We can generalize our result for R to an arbitrary shape:

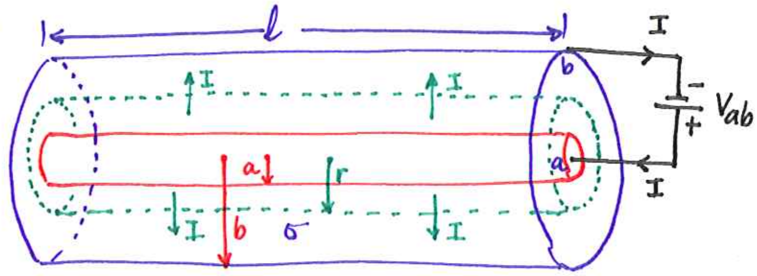
$$\boxed{R = \frac{V}{I} = \frac{-\int_l \vec{E}\cdot d\vec{l}}{\int_s \vec{J}\cdot d\vec{s}} = \frac{-\int_l \vec{E}\cdot d\vec{l}}{\int_s \sigma\vec{E}\cdot d\vec{s}}}$$

$\frac{1}{R}$ is called conductance G (Ω^{-1}), or siemens (S). For the linear resistor:

$$G = \frac{1}{R} = \sigma\frac{A}{l} \quad (S)$$

Example Conductance of Coaxial Cable:

Obtain an expression for G', the conductance per unit length of the insulation layer.



Since the current is radial, the area through which the current flows is $A = 2\pi r l$. Hence:

$$\vec{J} = \hat{r}\frac{I}{A} = \hat{r}\frac{I}{2\pi r l}$$

$$\vec{J} = \sigma\vec{E} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = \hat{r}\frac{I}{2\pi\sigma r l}$$

Current I flows from higher potential to lower potential:

$$V_{ab} = -\int_b^a \vec{E}\cdot d\vec{l} = -\int_b^a \frac{I}{2\pi\sigma l}\,\frac{\hat{r}\cdot\overbrace{\hat{r}\,dr}^{d\vec{l}}}{r} = \frac{I}{2\pi\sigma l}\int_a^b \frac{dr}{r} = \frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right)$$

The conductance per unit length is then:

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab}l} = \frac{I}{\frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right) l} = \frac{2\pi\sigma}{\ln\left(\frac{b}{a}\right)} \quad (S/m)$$

Joule's Law

Let's now consider the power dissipated in a conducting medium in the presence of an electrostatic field $\vec{E}$. The electric force acting on charges q_e and q_h are:

$$\vec{F}_e = q_e \vec{E} = \rho_{ve} \Delta V \vec{E}$$

ΔV is the element of volume.

$$\vec{F}_h = q_h \vec{E} = \rho_{vh} \Delta V \vec{E}$$

The energy (work) expended by electric fied in moving q_e by distance Δl_e is:

$$\Delta W = F_e \cdot \Delta l_e + F_h \cdot \Delta l_h$$

The power measured in watts (W) is:

$$\Delta P = \frac{\Delta W}{\Delta t} = \vec{F}_e \cdot \frac{\Delta \vec{l}_e}{\Delta t} + \vec{F}_h \cdot \frac{\Delta \vec{l}_h}{\Delta t}$$

$$= \vec{F}_e \cdot \vec{u}_e + \vec{F}_h \cdot \vec{u}_h$$

$$= \left(\rho_{ve} \vec{E} \cdot \vec{u}_e + \rho_{vh} \vec{E} \cdot \vec{u}_h\right) \Delta V$$

$$= \vec{E} \cdot \vec{J} \, \Delta V$$

So for a volume V, the total dissipated power is:

$$\boxed{P = \int_V \vec{E} \cdot \vec{J} \, dv} \quad (W) \quad (\text{Joul's law})$$

Since $\vec{J} = \sigma \vec{E} \Rightarrow P = \int_V \sigma |E|^2 dv$

$V = lA \Rightarrow$ Separating the integral: $P = \int_V \sigma |E|^2 dv = \int_A \sigma E_x ds \int_l E_x dl$

$$\Rightarrow \quad P = \underbrace{(\sigma E_x A)}_{JA} \underbrace{(E_x l)}_{V} = IV \quad (W)$$

using $V = IR \Rightarrow \quad P = RI^2 \quad (W)$